NON-STEADY FLOW OF RAREFIED GAS

(NEUSTANOVIVSHEESIA DVIZHENIE RAZREZHENNOGO GAZA)

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V.A. LEVIN
(Moscow)

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The nonsteady flow of a rarefied gas, which occurs when gaseous clouds undergo expansion and when gas flows out of a source, is discussed in this paper. Much material has been published on this particular problem. The motion which arises during the expansion of simple gas configurations within a continuous medium was investigated in [1 to 3]. From the gas-kinetic point of view, similar problems were discussed in [4], in which solutions were obtained, describing the collisionless expansion of gas clouds and the collisionless flow from a source and a jet. The expansion of symmetrical 'puffs' was also discussed in [5]. In [6] a solution for the expansion into a semi-infinite space was obtained. The problem about the rarefied gas flow from a source was solved in [7 and 8]. In all the above papers it was assumed that external fields of force were absent. Recently, a result was obtained for the expansion of a point mass of gas in a constant field [9].

The solution in [4, 6 and 9] uses Boltzmann's equation in which the collision integral is neglected as a basis to obtain the velocity distribution function. When external fields are absent, such a solution can be written down at once. In the presence of a constant field of force, the solution of the Boltzmann equation was obtained in [9]. Below, a group of similar problems involving the presence of various fields of force, is proposed to be solved by a general method. The effectiveness of this method is illustrated by a series of examples.

First we shall consider the motion of gas which, at the instant $t = \tau$, is concentrated in the neighborhood of a point in space $\mathbf{r}_0 = (x_1^\circ, x_2^\circ, x_3^\circ)$. The velocity distribution function for the particles at the given instant $f(\mathbf{u}^\circ, \tau)$ is known and there is also an external field of force. Then, the motion of gas will be described by the following system of equations, using Lagrange variables

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} = \mathbf{F}(\mathbf{r}, \mathbf{u}, t), \qquad \frac{\partial \mathbf{r}}{\partial t} = \mathbf{u}, \qquad n\frac{\partial (x_1, x_2, x_3)}{\partial (u_1^\circ, u_2^\circ, u_3^\circ)} = f(\mathbf{u}^\circ, \tau) \tag{1}$$

where the time t and the initial velocity of the corresponding particles are taken as the Lagrange coordinates. In equations of motion the functions F_i will be known functions of their arguments, and the last equation in (1) is the equation of continuity, in Lagrange variables [10]. It follows, that to determine the whole flow pattern we must integrate the system (1) with the initial conditions $\mathbf{r} = \mathbf{r}_0$ and $\mathbf{u} = \mathbf{u}^0$ when $t = \tau$. Having solved this Cauchy problem, we shall find a parametric expression for velocity and density in terms of initial velocity. Excluding the initial velocity from the solution, we shall obtain the velocity and the density distribution at each point in space as functions of time.

It is necessary to note that this approach to the solution of the problem will be true so long as the Jacobian of the transformation $d\,(x_1, x_2, x_3;\, u_1^\circ, u_2^\circ, u_3^\circ) \neq 0$ differs from zero. Vanishing of the Jacobian would imply that the particles which left the point with diffiverent initial velocities could be found occupying the same position in space, and from this instant, our solution would cease to be true.

If on the other hand the gas occupied some region D at the initial instant, then the corresponding solution would be given by the formulas

$$N = \iiint\limits_D n\,(\mathbf{r}, \mathbf{r}_0, t, \tau)\, d\mathbf{r}_0, \qquad \mathbf{U} = \frac{1}{N}\iiint\limits_D n\mathbf{u}\,d\mathbf{r}_0, \quad (d\mathbf{r}_0 = dx_1^\circ dx_2^\circ dx_3^\circ) \tag{2}$$

If T is the time of performance ofthe distributed source, then the solution in this case has the form:

$$N = \int\limits_0^T \iiint\limits_D H\,(t-\tau)\, n\, d\mathbf{r}_0\, d\tau$$

$$\mathbf{U} = \frac{1}{N}\int\limits_0^T \iiint\limits_D H\,(t-\tau)\, n\mathbf{u}\, d\mathbf{r}_0\, d\tau, \qquad H\,(t-\tau) = \begin{cases} 0 & (t < \tau) \\ 1 & (t > \tau) \end{cases} \tag{3}$$

In the formulas of (2.3) the values n and u are the solution of the problem concerning the motion of a point mass of gas, which at the instant $t = \tau$ is located at the point r_0 with an initial velocity distribution of the particles $f\,(\mathbf{r}_0, \mathbf{u}^\circ, \tau)$. Knowing the mean gas velocity and the velocities from various points, it is possible to calculate the pressure tensor, the temperature and the higher moments. Thus, for the pressure tensor we obtain

$$P_{ij}\,(\mathbf{r}, t) = \iiint\limits_D m\,(u_i - U_i)\,(u_j - U_j)\, n d\mathbf{r}_0 \qquad (P_{ij} = P_{ij} - p\delta_{ij},\ 3p = P_{11} + P_{22} + P_{33}) \tag{4}$$

where we have the usual notation. For the temperature, we have

$$T = \frac{2}{3kN}\iiint\limits_D \frac{mn}{2}\sum_{i=1}^{3} (u_i - U_i)^2\, d\mathbf{r}_0 \tag{5}$$

From the expression for hydrostatic pressure it follows that $p = NkT$. It is not difficult to verify that the hydrodynamic equations for the whole of the gas are satisfied identically.

We shall consider several examples.

1. *Expansion of a gas cloud in a resisting medium.* For the expansion of a point mass of gas, the system of equations has the form:

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} = -\nu\frac{\partial \mathbf{r}}{\partial t}, \qquad \frac{\partial \mathbf{r}}{\partial t} = \mathbf{u}$$

$$n\,\frac{\partial\,(x_1, x_2, x_3)}{\partial\,(u_1^\circ, u_2^\circ, u_3^\circ)} = n_0\left(\frac{\beta}{\pi}\right)^{3/2} \exp\,[-\beta\,(u_1^{\circ 2} + u_2^{\circ 2} + u_3^{\circ 2})] \quad \left(\beta = \frac{m}{2kT_0}\right) \tag{6}$$

where m is the mass of a particle, k is Boltzmann's constant, T_0 is the temperature, and ν is the collision frequency per single particle. The initial distribution function is assumed to be Maxwellian, although this need not be the case. Solving (6) and excluding the initial velocities, we obtain

$$\mathbf{u} = \frac{\nu(\mathbf{r}-\mathbf{r}_0)e^{-\nu t}}{1-e^{-\nu t}}, \quad n = n_0\nu^3\left(\frac{\beta}{\pi}\right)^{3/2}(1-e^{-\nu t})^{-3}\exp\left[-\frac{\beta\nu^2}{(1-e^{-\nu t})^2}\sum_{i=1}^{3}(x_i-x_i^\circ)^2\right] \tag{7}$$

When $t\to\infty$, we obtain a quiescent gas whose limiting density distribution is

$$n_\infty = n_0\nu^3\left(\frac{\beta}{\pi}\right)^{3/2}\exp\left[-\beta\nu^2\sum_{i=1}^{3}(x_i-x_i^\circ)^2\right] \tag{8}$$

Assumption of the friction tending to zero $\nu\to 0$ in the expressions (7) results in a known solution [4 and 9] describing the expansion of a point mass of gas in free space.

Using the expressions (2) for the mean velocity and density, we obtain the formulas for the expansion of a volume D

$$N = n_0\left(\frac{\beta}{\pi}\right)^{3/2}\nu^3\iiint_D(1-e^{-\nu t})^{-3}\exp\left[-\frac{\beta\nu^2}{(1-e^{-\nu t})^2}\sum_{i=1}^{3}(x_i-x_i^\circ)^2\right]d\mathbf{r}_0$$

$$\mathbf{U} = n_0\nu^4\left(\frac{\beta}{\pi}\right)^{3/2}\frac{e^{-\nu t}}{N}\iiint_D(\mathbf{r}-\mathbf{r}_0)(1-e^{-\nu t})^{-4}\exp\left[-\frac{\beta\nu^2}{(1-e^{-\nu t})^2}\sum_{i=1}^{3}(x_i-x_i^\circ)^2\right]d\mathbf{r}_0 \tag{9}$$

If the gas initially filled a semi-infinite space, a two-dimensional layer, a cylindrical cavity, or a sphere, then putting $\nu\to 0$ in (9) and carrying out a simple integration, results in known formulas describing collisionless expansion of the corresponding regions of gas [4 to 6].

2. *Collisionless source in a field of force* (fig. 1). Let a point source of gas be placed at the origin of a coordinate system in presence of a constant field of force (the force of gravity). We also assume that the source strength I (for simplicity taken as constant) and the velocity distribution function for the emitted particles are given. We shall take the distribution function in the form:

$$f(\mathbf{u}^\circ,\tau) = I\left(\frac{\beta}{\pi}\right)^{3/2}\exp\{-\beta[u_1^{\circ 2}+(u_2^\circ-V_2)^2+(u_3^\circ-V_3)^2]\} \tag{10}$$

i.e. the particles are emitted with a mean velocity $V=(0, V_2, V_3)$. Leaving out the intermediate steps, we obtain

$$N = I\left(\frac{\beta}{\pi}\right)^{3/2}\int_0^T\frac{H(t-\tau)}{(t-\tau)^3}\times$$

$$\times\exp\left\{-\beta\frac{x_1^2+[x_2-V_2(t-\tau)]^2+[x_3-V_3(t-\tau)+{}^1\!/_2\,g(t-\tau)^2]^2}{(t-\tau)^2}\right\}d\tau$$

$$U_i = \frac{1}{N}I\left(\frac{\beta}{\pi}\right)^{3/2}\int_0^T\frac{H(t-\tau)}{(t-\tau)^4}x_i\times$$

$$\times\exp\left\{-\beta\frac{x_1^2+[x_2-V_2(t-\tau)]^2+[x_3-V_3(t-\tau)+{}^1\!/_2\,g(t-\tau)^2]^2}{(t-\tau)^2}\right\}dt \qquad (i=1,2) \tag{11}$$

$$U_3 = \frac{1}{N} I \left(\frac{\beta}{\pi}\right)^{3/2} \int_0^T \frac{H(t-\tau)}{(t-\tau)^4} [x_3 - {}^1/_2\, g(t-\tau)^2] \times$$

$$\times \exp\left\{-\beta \frac{x_1^2 + [x_2 - V_2(t-\tau)]^2 + [x_3 - V_3(t-\tau) + {}^1/_2\, g(t-\tau)^2]^2}{(t-\tau)^2}\right\} d\tau$$

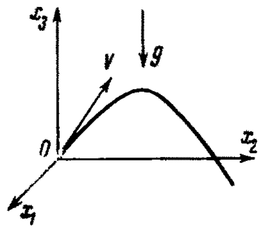

FIG. 1

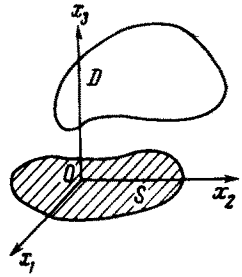

FIG. 2

Assuming the source to work for an infinite time $T = \infty$, we obtain from these formulas expressions for a point source when $V_2 = V_3 = g = 0$, but expressions for a collisonless jet when $V_2 = g = 0$, $V_3 \neq 0$ [4].

3. *Nonsteady rarefied gas flow around a flat plate.* Let the gas occupy a region D in the semi-infinite space $x_3 > 0$ at the initial instant of time. Maxwellian distribution function is assumed and a flat plate (fig. 2) is placed in the plane $x_3 = 0$.

The expanding gas begins to flow around the plate. For the solution of the problem we must find the law of interaction between the incident particles and the plate. We will suppose that the incident particles are scattered diffusely, i.e. at the plate we shall assume the distribution function for the reflected particles to be [11]

$$f(\mathbf{u}^\circ, x_1, x_2, \tau) = \frac{2}{\pi\beta_0^2} I(x_1, x_2, \tau) \exp[-\beta_0 (u_1^{\circ 2} + u_2^{\circ 2} + u_3^{\circ 2})] \qquad (12)$$

Where $I(x_1, x_2, \tau)$ is the flux of particles incident on the plate.

The motion of the gas during the expansion of the cloud in the empty space is given by the formulas

$$N = n_0 \left(\frac{\beta}{\pi}\right)^{3/2} \iiint_D \frac{1}{t^3} \exp\left[-\frac{\beta}{t^2} \sum_{i=1}^3 (x_i - x_i^\circ)^2\right] d\mathbf{r}_0$$

$$N\mathbf{U} = n_0 \left(\frac{\beta}{\pi}\right)^{3/2} \iiint_D \frac{\mathbf{r} - \mathbf{r}_0}{t^4} \exp\left[-\frac{\beta}{t^2} \sum_{i=1}^3 (x_i - x_i^\circ)^2\right] d\mathbf{r}_0 \qquad (13)$$

For the incident flux of particles on the plate we have

$$I(x_1, x_2, \tau) = n_0 \left(\frac{\beta}{\pi}\right)^{3/2} \iiint_D \frac{x_3^\circ}{\tau^4} \exp\left[-\beta \frac{(x_1 - x_1^\circ)^2 + (x_2 - x_2^\circ)^2 + x_3^{\circ 2}}{\tau^2}\right] d\mathbf{r}_0 \qquad (14)$$

Thus it appears that at the plate there is a distributed source with a known distribution function for the scattered particles. Assuming β_0 constant, we will find a formula describing the flow of the scattered particles in the form:

$$N_* = \frac{2}{\pi\beta_0^2} \int_0^\infty \frac{H(t-\tau)}{(t-\tau)^3} \iint_S I(\xi_1, \xi_2, \tau) \exp\left[-\frac{\beta_0}{(t-\tau)^2} \sum_{i=1}^3 (x_i - \xi_i)^2\right] d\xi_1 d\xi_2 d\tau \qquad (15)$$

$$N_*\mathbf{U}_* = \frac{2}{\pi\beta_0^2}\int_0^\infty \frac{H(t-\tau)}{(t-\tau)^4}\iint_S (\mathbf{r}-\xi)\, I(\xi_1, \xi_2, \tau) \times$$
$$\times \exp\left[-\frac{\beta_0}{(t-\tau)^2}\sum_{i=1}^3 (x_i - \xi_i)^2\right] d\xi_1,\, d\xi_2,\, d\tau \qquad \xi = (\xi_1, \xi_2, 0)$$

The integration in (15) is performed over the surface of the plate. The mean density of the incident and reflected gas will be found in the form of the sum of the densities $N^* = N + N_*$, and the mean velocity by

$$\mathbf{U}^* = (N\mathbf{U} + N_*\mathbf{U}_*)/N^*.$$

In order to find the flow behind the plate (13) must be integrated, but not over the whole region D, but only over that part of the initial volume, which is seen from the given point placed behind the plate. The distributions of temperature, pressure and other higher moments can also be determined, if required.

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